MATH 3E HOMEWORK II

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Problem I

Suppose that A is an $n \times n$ matrix, with $A = LU$ its decomposition into a product of lower and upper triangular matrices. Show that this decomposition is unique (i.e. if $A = L'U'$ is a decomposition of A into product of lower and upper triangular matrices, then $L = L'$ and $U = U'$).

Answer.

Proof. Since $A = LU = L'U'$,

$$\begin{aligned} U\left(U'\right)^{-1} &= \left(L^{-1}L\right)U\left(U'\right)^{-1} \\ &= L^{-1}\left(LU\right)\left(U'\right)^{-1} \\ &= L^{-1}\left(L'U'\right)\left(U'\right)^{-1} \\ &= \left(L^{-1}L'\right)U'\left(U'\right)^{-1} \\ &= L^{-1}L' \end{aligned}$$

However, $U\left(U'\right)^{-1}$ is upper triangular and $L^{-1}L'$ is lower triangular. Therefore there must be a diagonal matrix $L^{-1}L'$ with 1s on the main diagonal. Hence, $U\left(U'\right)^{-1} = L^{-1}L' = I$. Therefore $L = L'$ and $U = U'$. □

Problem II

Find the LU decomposition for

$$\begin{bmatrix} 2 & -2 & 6 & -4 \\ 2 & -5 & 2 & 2 \\ -4 & 1 & 3 & 2 \\ 1 & 5 & 1 & -2 \end{bmatrix}$$

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Answer. Let the given function A, then

$$A=\begin{bmatrix} 2 & -2 & 6 & -4 \\ 2 & -5 & 2 & 2 \\ -4 & 1 & 3 & 2 \\ 1 & 5 & 1 & -2 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{bmatrix}$$

$$A_1=\begin{bmatrix} 2 & -2 & 6 & -4 \\ 0 & -3 & -4 & 6 \\ 0 & -3 & 15 & -6 \\ 0 & 6 & -2 & 0 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ -2 & * & 1 & \\ \frac{1}{2} & * & * & 1 \end{bmatrix}$$

$$A_2=\begin{bmatrix} 2 & -2 & 6 & -4 \\ 0 & -3 & -4 & 6 \\ 0 & 0 & 19 & -12 \\ 0 & 0 & -10 & 12 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ -2 & 1 & 1 & \\ \frac{1}{2} & -2 & * & 1 \end{bmatrix}$$

$$\underline{A_3=U=\begin{bmatrix} 2 & -2 & 6 & -4 \\ 0 & -3 & -4 & 6 \\ 0 & 0 & 19 & -12 \\ 0 & 0 & 0 & \frac{108}{19} \end{bmatrix}, \quad L=\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ -2 & 1 & 1 & \\ \frac{1}{2} & -2 & -\frac{10}{19} & 1 \end{bmatrix}}$$

Problem III

Assume B, C, and D are invertible. Compute A^{-1}, A^T, and $\left(A^T\right)^{-1}$ where $A=3B^T\left(C^T\right)^3D^2$

Answer.

$$\begin{aligned} A^{-1} =& \frac{1}{3}\left[B^T\left(C^T\right)^3D^2\right]^{-1} \\ =& \frac{1}{3}\left(D^2\right)^{-1}\left[B^T\left(C^T\right)^3\right]^{-1} \\ =& \underline{\frac{1}{3}\left(D^{-1}\right)^2\left[\left(C^T\right)^{-1}\right]^3\left(B^T\right)^{-1}} \end{aligned}$$

$$\begin{aligned} A^T =& 3\left[B^T\left(C^T\right)^3D^2\right]^T \\ =& 3\left(D^2\right)^T\left[B^T\left(C^T\right)^3\right]^T \\ =& \underline{3\left(D^T\right)^2C^3B} \end{aligned}$$

$$\begin{aligned}\left(A^T\right)^{-1} =& \frac{1}{3}\left[\left(D^T\right)^2 C^3 B\right]^{-1}\\ =& \frac{1}{3}B^{-1}\left[\left(D^T\right)^2 C^3\right]^{-1}\\ =& \underline{\frac{1}{3}B^{-1}\left(C^{-1}\right)^3\left[\left(D^T\right)^{-1}\right]^2}\end{aligned}$$

Problem IV

A matrix A is said to be skew symmetric iff $A^T = -A$. Show

For a square matrix B, $B - B^T$ is skew symmetric.

Answer.

Proof.

$$\begin{aligned}\left(B - B^T\right)^T =& B^T - B\\ =& -\left(B - B^T\right)\end{aligned}$$

□

Every diagonal element of a skew symmetric matrix is 0.

Answer.

Proof. Let A:

$$A = \begin{bmatrix} a_{1,1} & \cdots & \cdots & a_{1,n} \\ \vdots & a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & \cdots & a_{n,n} \end{bmatrix} \quad \{a_{i,j} \mid i = \{1,\,2,\,\cdots,\,n-1,\,n\};\ j = \{1,\,2,\,\cdots,\,n-1,\,n\}\}$$

then

$$A^T = -A$$

$$\Rightarrow A^T + A = \begin{bmatrix} a_{1,1} & \cdots & \cdots & a_{n,1} \\ \vdots & a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & \cdots & \cdots & a_{n,n} \end{bmatrix} + \begin{bmatrix} a_{1,1} & \cdots & \cdots & a_{1,n} \\ \vdots & a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & \cdots & a_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} + a_{1,1} & \cdots & \cdots & a_{n,1} + a_{1,n} \\ \vdots & a_{2,2} + a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} + a_{n,1} & \cdots & \cdots & a_{n,n} + a_{n,n} \end{bmatrix}$$

$$= \mathbf{0}$$

$$\Rightarrow \{\forall i = j\} \to \{a_{i,j} + a_{i,j}\} + = \{a_{i,j}\} = 0$$

□

Problem V

Exercises §1.5.34 and §1.5.35 in Hill.

Answer §1.5.34.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -1 \\ & 3 & -2 \\ & & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 29 \\ 7 \\ -13 \end{bmatrix}$$

$$PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 29 \\ 7 \\ -13 \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \\ 29 \end{bmatrix}$$

Let $UX = Y \, (= (y_1, \, y_2, \, y_3))$, then we obtain Y by solving $LY = PB$

$$\begin{bmatrix} 1 & & & 7 \\ -2 & 1 & & -13 \\ 3 & 2 & 1 & 29 \end{bmatrix} \to \begin{bmatrix} 1 & & & 7 \\ 0 & 1 & & 1 \\ 0 & 2 & 1 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & & & 7 \\ 0 & 1 & & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$\Rightarrow (y_1,\, y_2,\, y_3) = (7,\, 1,\, 6)$$

Thus we obtain X by solving $UX = Y$

$$\begin{bmatrix} 2 & -3 & -1 & 7 \\ & 3 & -2 & 1 \\ & & -3 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -3 & 8 \\ & 3 & -2 & 1 \\ & & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 2 \\ & 3 & 0 & -3 \\ & & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & -1 \\ & & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \underline{X = (x_1,\, x_2,\, x_3) = (1,\, -1,\, -2)}$$

Answer §1.5.35.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & -1 & 0 \\ & -2 & -2 \\ & & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -8 \\ 10 \\ -7 \end{bmatrix}$$

$$PB = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -8 \\ 10 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ -8 \\ 10 \end{bmatrix}$$

Let $UX = Y\ (= (y_1,\, y_2,\, y_3))$, then we obtain Y by solving $LY = PB$

$$\begin{bmatrix} 1 & & & -7 \\ 2 & 1 & & -8 \\ -3 & -2 & 1 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & & & -7 \\ 0 & 1 & & 6 \\ 0 & -2 & 1 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & & & -7 \\ 0 & 1 & & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow Y = (y_1,\, y_2,\, y_3) = (-7,\, 6,\, 1)$$

Thus we obtain X by solving $UX = Y$

$$\begin{bmatrix} -3 & -1 & 0 & -7 \\ & -2 & -2 & 6 \\ & & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -3 & -1 & 0 & -7 \\ & -2 & 0 & 4 \\ & & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 0 & -9 \\ & 1 & 0 & -2 \\ & & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ & 1 & 0 & -2 \\ & & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \underline{X = (x_1,\, x_2,\, x_3) = (3,\, -2,\, -1)}$$

Problem VI

Find a 2×2 matrix A such that $A^2 = -I$ (A having only real entries).

Answer. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then

$$A^2 = -I$$

$$\Rightarrow \begin{bmatrix} a^2 + bc & b\,(a+d) \\ c\,(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a^2 + bc = bc + d^2 = -1 & (1) \\ b\,(a+d) = c\,(a+d) = 0 & (2) \end{cases}$$

From (2), $b = c = 0$ or $(a+d) = 0$.

Case 1: If $b = c = 0$, then

$$(1) \Rightarrow a^2 + bc = bc + d^2 = -\,1$$

$$\Rightarrow a^2 = d^2 = -1$$

$$a = d = \boldsymbol{i}\,(\notin \mathbb{R})$$

Therefore $b \neq 0$ and $c \neq 0$ since $a \in \mathbb{R}$ and $d \in \mathbb{R}$.

Case 2: If $(a+d) = 0$, then

$$(1) \Rightarrow a^2 + bc = bc + d^2 = -\,1$$

$$\Rightarrow a^2 = d^2 = -1 - bc$$

$$\Rightarrow \begin{cases} a = d = 0 & \left(\because a + d = 0,\, a^2 = d^2\right) & (4) \\ -1 - bc = 0 & \end{cases}$$

$$\Rightarrow c = -\frac{1}{b} \qquad (5)$$

From (4) and (5), $\underline{A = \begin{bmatrix} 0 & b \\ -\frac{1}{b} & 0 \end{bmatrix}; \quad \{\forall b \in \mathbb{R}\}}$